

Probability representation and state-extended uncertainty relations

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Abstract

The new inequality recently found by Trifonov and called the state-extended inequality is considered in the tomographic-probability representation of quantum mechanics. The Trifonov uncertainty relations are expressed in terms of optical tomograms and can be checked in experiments on homodyne detection of the photon states.

Keywords: uncertainty relations, probability distribution, quantum tomography, optical tomograms.

1 Introduction

Nonclassical states of photons [1] are the subject of investigation in homodyne detecting experiments. The uncertainty relations [2], being the basic principles of quantum mechanics, can be presented in different forms. Recently [3] a new formulation of quantum mechanics, called the tomographic-probability representation of quantum mechanics, was introduced. The tomographic approach provides the possibility to express all quantum postulates and equations for the wave function and density matrix in the form where the fair probability distributions are used instead of the density matrices (wave functions). A new modification of the uncertainty relations was found by Trifonov [4]. These relations have not yet been checked experimentally. In this paper, we present the Trifonov uncertainty relations in the tomographic form, which nowadays is the most appropriate method for checking experimentally the basic principles of quantum mechanics by homodyne detection of the photon states.

2 State-Extended Uncertainty Relations

The standard uncertainty relation of Heisenberg [2] for the position and momentum looks as follows:

$$\sigma_{qq}\sigma_{pp} \geq 1/4. \quad (1)$$

We take the Planck's constant $\hbar = 1$. For the pure state $|\Psi\rangle$, (1) reads

$$\sigma_{qq} = \text{Tr} \left(\hat{q}^2 |\Psi\rangle\langle\Psi| \right) - (\text{Tr} (\hat{q} |\Psi\rangle\langle\Psi|))^2, \quad \sigma_{pp} = \text{Tr} \left(\hat{p}^2 |\Psi\rangle\langle\Psi| \right) - (\text{Tr} (\hat{p} |\Psi\rangle\langle\Psi|))^2, \quad (2)$$

which are dispersions of the position and momentum, respectively.

Trifonov [4] proved that this relation can be generalized for two different states $|\Psi_1\rangle$ and $|\Psi_2\rangle$. This generalization, called the state-extended uncertainty relation, is

$$\begin{aligned} & \frac{1}{2} \left[\text{Tr} \left(\hat{q}^2 |\Psi_1\rangle \langle \Psi_1| \right) - (\text{Tr} (\hat{q} |\Psi_1\rangle \langle \Psi_1|))^2 \right] \left[\text{Tr} \left(\hat{p}^2 |\Psi_2\rangle \langle \Psi_2| \right) - (\text{Tr} (\hat{p} |\Psi_2\rangle \langle \Psi_2|))^2 \right] \\ & + \frac{1}{2} \left[\text{Tr} \left(\hat{q}^2 |\Psi_2\rangle \langle \Psi_2| \right) - (\text{Tr} (\hat{q} |\Psi_2\rangle \langle \Psi_2|))^2 \right] \left[\text{Tr} \left(\hat{p}^2 |\Psi_1\rangle \langle \Psi_1| \right) - (\text{Tr} (\hat{p} |\Psi_1\rangle \langle \Psi_1|))^2 \right] \geq \frac{1}{4}. \end{aligned} \quad (3)$$

Inequality (3) provides constraints for dispersions of the position and momentum. The difference between the Trifonov inequality (3) and the Heisenberg uncertainty relation (1) consists in the fact that two states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are involved in (3), while in the Heisenberg inequality (1), only one state $|\Psi\rangle$ is taken into account. One can see that, if in (3) we take $|\Psi_1\rangle = |\Psi_2\rangle$, inequality (3) provides the Heisenberg uncertainty relation as a particular case of inequality (3).

Now we pose the question — is it possible to check experimentally the state-extended uncertainty relations (3)? For such a check, we suggest to use homodyne detection of the photon states. At the output of this experiment, one obtains the so-called optical tomogram, which contains complete information on the photon quantum state.

3 Optical Tomography of Photon States

In this section, we review the approach called optical tomography of quantum states where the fair probability-distribution function $w(X, \theta)$ is used as an alternative to the wave function and density matrix [3, 5, 6, 7]. The probability-distribution function $w(X, \theta)$ is called the optical tomogram of the quantum state. The argument X of the function is called the homodyne quadrature of the photon, and the angle θ is called the local oscillator phase — and this parameter is controlled in experiments on homodyne measuring of the quantum states [8, 9, 10]. The optical tomogram is related to the wave function $\Psi(y)$ in view of the formula [11]

$$w(X, \theta) = \frac{1}{2\pi |\sin \theta|} \left| \int \Psi(y) \exp \left(\frac{iy^2}{2 \tan \theta} - \frac{iXy}{\sin \theta} \right) dy \right|^2. \quad (4)$$

For the local oscillator phase $\theta = 0$, the optical tomogram provides the probability distribution of the position, i.e.,

$$w(X, 0) = |\Psi(x)|^2. \quad (5)$$

For the local oscillator phase $\theta = \pi/2$, the optical tomogram provides the probability distribution of the momentum, i.e.,

$$w(X, \pi/2) = |\tilde{\Psi}(X)|^2, \quad (6)$$

where $\tilde{\Psi}(X)$ is the quantum-state wave function $|\Psi\rangle$ in the momentum representation, i.e.,

$$\tilde{\Psi}(X) = \frac{1}{\sqrt{2\pi}} \int \Psi(y) e^{iXy} dy. \quad (7)$$

Thus, one can obtain dispersions of the position and momentum if the tomogram $w(X, \theta)$ is used as a probability distribution, namely,

$$\text{Tr}(\hat{q}^2 |\Psi\rangle\langle\Psi|) = \int w(X, 0) X^2 dX, \quad (8)$$

$$\text{Tr}(\hat{p}^2 |\Psi\rangle\langle\Psi|) = \int w(X, \pi/2) X^2 dX. \quad (9)$$

In view of (8) and (9), the Heisenberg uncertainty relation can be rewritten in the tomographic form (see e.g., [12, 6, 13]) as follows:

$$\left[\int w(X, 0) X^2 dX - \left(\int w(X, 0) X dX \right)^2 \right] \left[\int w(X, \pi/2) X^2 dX - \left(\int w(X, \pi/2) X dX \right)^2 \right] \geq \frac{1}{4}. \quad (10)$$

Formula (10) is the tomographic form of the Heisenberg inequality (1).

4 Trifonov Inequality in the Tomographic Representation

Inequality (3) can be presented in the tomographic form. For this, we express dispersions of the position and momentum in the first state $|\Psi_1\rangle$ and the second state $|\Psi_2\rangle$ in terms of optical tomograms $w_1(X, \theta)$ and $w_2(X, \theta)$ of these states. The expressions for the position dispersions read

$$\langle \Psi_1 | \hat{q}^2 | \Psi_1 \rangle - (\langle \Psi_1 | \hat{q} | \Psi_1 \rangle)^2 = \int w_1(X, 0) X^2 dX - \left(\int w_1(X, 0) X dX \right)^2 \quad (11)$$

and

$$\langle \Psi_2 | \hat{q}^2 | \Psi_2 \rangle - (\langle \Psi_2 | \hat{q} | \Psi_2 \rangle)^2 = \int w_2(X, 0) X^2 dX - \left(\int w_2(X, 0) X dX \right)^2. \quad (12)$$

The dispersions of the momentum in the first and second states are

$$\langle \Psi_1 | \hat{p}^2 | \Psi_1 \rangle - (\langle \Psi_1 | \hat{p} | \Psi_1 \rangle)^2 = \int w_1(X, \pi/2) X^2 dX - \left(\int w_1(X, \pi/2) X dX \right)^2 \quad (13)$$

and

$$\langle \Psi_2 | \hat{p}^2 | \Psi_2 \rangle - (\langle \Psi_2 | \hat{p} | \Psi_2 \rangle)^2 = \int w_2(X, \pi/2) X^2 dX - \left(\int w_2(X, \pi/2) X dX \right)^2. \quad (14)$$

We can introduce the dispersions in a rotated reference frame. In fact, the rotated quadrature operator

$$\hat{X} = \hat{q} \cos \theta + \hat{p} \sin \theta \quad (15)$$

has the physical meaning of the position in the rotated reference frame for the local oscillator phase θ and the physical meaning of the momentum in the rotated reference frame for the local oscillator phase $\theta + \pi/2$. In view of this, we can write the Trifonov inequality not only for the position and momentum in the initial reference frame but obtain the inequality in the rotated reference frame.

For this, we replace in (11)–(14) the zero local oscillator phase by the local oscillator phase θ . As a result, we arrive at the inequality

$$\begin{aligned} & \frac{1}{2} \left[\int w_1(X, \theta) X^2 dX - \left(\int w_1(X, \theta) X dX \right)^2 \right] \\ & \times \left[\int w_2(X, \theta + \pi/2) X^2 dX - \left(\int w_2(X, \theta + \pi/2) X dX \right)^2 \right] \\ & + \frac{1}{2} \left[\int w_2(X, \theta) X^2 dX - \left(\int w_2(X, \theta) X dX \right)^2 \right] \\ & \times \left[\int w_1(X, \theta + \pi/2) X^2 dX - \left(\int w_1(X, \theta + \pi/2) X dX \right)^2 \right] \geq 1/4. \end{aligned} \quad (16)$$

This inequality at $\theta = 0$ provides the Trifonov inequality in the tomographic form. Inequality (16) can be checked in experiments where two optical tomograms for two different photon states are measured. In fact, inequality (16) provides a criterium for checking the accuracy of the experiments with homodyne detection of photon states, since for arbitrary different pairs of optical tomograms this inequality must be fulfilled. In the classical domain, this inequality can be violated. Relation (16) can be easily extended to tomograms corresponding to mixed quantum states.

For the pure state, the tomogram $w_1(X, \mu, \nu)$ determined by the optical tomogram $w_1(X, \theta)$, in view of the formula

$$w_1(X, \mu, \nu) = \frac{1}{\sqrt{\mu^2 + \nu^2}} w_1 \left(\frac{X}{\sqrt{\mu^2 + \nu^2}}, \arctan(\nu/\mu) \right), \quad (17)$$

satisfies the integral equality

$$\frac{1}{2\pi} \int w_1(X, \mu, \nu) w_1(Y, \mu, \nu) e^{i(X-Y)} dX dY d\mu d\nu = 1. \quad (18)$$

The same relation is valid for the pure-state tomogram $w_2(X, \theta)$. Inequality (16) is correct if tomograms $w_1(X, \theta)$ and $w_2(X, \theta)$ correspond to mixed states. This means that they are the convex sums of tomograms satisfying equality (18). For mixed states, the tomograms provide a value of integral (18) smaller than 1. Thus, the tomographic form of the Trifonov inequality can be checked experimentally not only for the pure quantum states but also for mixed quantum states.

5 Conclusions

To conclude, we point out the main results of this work.

We presented the state-extended uncertainty relations for the position and momentum in the tomographic form. This form contains only experimental optical tomograms that can be obtained by homodyne detection of the photon states. We formulated the state-extended inequality for the position and momentum and also found the generalized inequality that is valid for an arbitrary local oscillator phase. We suggest to use the tomographic state-extended uncertainty relation obtained as a criterion for checking the accuracy of homodyne measurements of the photon quantum states. We propose to make an extra analysis of the homodyne experiments such as those performed in [8, 9, 10, 13, 14, 15] to check the state-extended inequality. One can easily write the multimode state-extended uncertainty relations [16] in the tomographic form, and we will do this in future papers.

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